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Truncation and Residualization with Weighted Balanced Coordinates

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Frequency-weighted internally balanced (FWIB) truncation is briefly reviewed. A previous frequency response error analysis for FWIB truncation is extended, and an exact error bound for the special case of order reduction by one state is presented in terms of the controllability-observability measure used in selecting the coordinate to truncate, as well as two additional frequency-dependent variables. An approximate error bound for the general case of order reduction by more than one state is also developed, under the assumption that only states with small controllability-observability measures are truncated. The error bound is shown to give justification for the FWIB truncation technique based only upon consideration of the controllability-observability measures. However, the presence of the two additional variables suggests a new technique based upon the derived criterion. FWIB residualization is presented, and a frequency response error analysis yields results similar to that found for FWIB truncation. Numerical examples are given to support the error analysis results, as well as to stress that FWIB truncation and residualization can be used in a coordinated manner to achieve higher accuracy than that achievable from either technique used alone.

Introduction

MODELS developed from governing physical principles are often of high dynamic order,[1,2] complicating the direct use of the model in the intended application. For example, control law synthesis is a common application for dynamic models. However, many modern linear control synthesis techniques produce a controller with dynamic order at least equal to the plant dynamic order.[3,4] This is unacceptable for controller implementation. Thus, order reduction of dynamic models is of extreme importance.

A reduced-order model for the plant, or for the compensator, should preserve key frequency-domain characteristics of the higher order model,[5–11] and an order reduction technique specifically tailored for this task is frequency-weighted internally balanced (FWIB) truncation.[5] In this technique, coordinates reflecting small measures of weighted controllability-observability are truncated based upon the argument that the procedure should yield a reduced-order model that matches the frequency response of the higher order model in the desired frequency range. Numerous examples have validated the algorithm.[8,12–14]

As yet, however, no rigorous theoretical result exists for this technique that guarantees a small error in the critical frequency range, similar to the result discovered for unweighted internally balanced (IB) truncation.[5,15] Special weighting filters have been developed that make the IB bound applicable to the weighted case.[16] However, these weightings are specifically tailored to yield the error bound result, not to obtain an accurate approximation in the desired frequency range. Enns[5] did consider a frequency response error analysis for his technique, but his result was left in a state with limited utility. The first goal of this paper is to extend the frequency response error analysis of Ref. 5 for FWIB truncation, so as to develop a theoretical justification for this weighted reduction technique.[11,17]

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Recall from classical order reduction theory that truncation is most appropriate for eliminating lower frequency dynamics, whereas residualization is more appropriate for eliminating higher frequency dynamics, relative to the frequency range of interest.[13,14] References 18 and 19 recently considered this point and showed that, in specific applications, IB reduction based upon residualization is better suited than IB truncation. Further, all the properties existing for IB truncation carried over to IB residualization. In light of these results, the second goal of this paper is to establish FWIB residualization as an appropriate order reduction technique, to be used in conjunction with FWIB truncation.[11,17]

Truncation with FWIB Coordinates

In this section, FWIB truncation is briefly reviewed. Consider a finite dimensional, linear, time-invariant state-space model representing the higher order system, or

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t) \tag{1}$$

Also consider input and output weighting filters

$$\begin{aligned} \dot{x}_{wi}(t) &= A_{wi}x_{wi}(t) + B_{wi}\delta(t) \\ \dot{x}_{wo}(t) &= A_{wo}x_{wo}(t) + B_{wo}y(t) \\ u(t) &= C_{wi}x_{wi}(t) + D_{wi}\delta(t) \\ \gamma(t) &= C_{wo}x_{wo}(t) + D_{wo}y(t) \end{aligned} \tag{2}$$

cascaded with the higher order model in Eq. (1). The input weighting filter is used to adjust the frequency response such that $\delta(j\omega)$ to $y(j\omega)$ is approximately the same as $u(j\omega)$ to $y(j\omega)$ within the frequency range of interest and is well attenuated outside the frequency range of interest. On the other hand, the output weighting filter can be used to adjust the frequency response such that $u(j\omega)$ to $\gamma(j\omega)$ is approximately the same as $u(j\omega)$ to $y(j\omega)$ in the desired region and is well attenuated otherwise.

The weighted controllability and observability grammians X and Y are defined as[5,8]

$$\begin{gathered} X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{X}(j\omega)\bar{X}^*(j\omega)\,\mathrm{d}\omega \\ \bar{X}(j\omega) = (j\omega I - A_X)^{-1}B_X \\ Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{Y}^*(j\omega)\bar{Y}(j\omega)\,\mathrm{d}\omega \\ \bar{Y}(j\omega) = C_Y(j\omega I - A_Y)^{-1} \\ A_X = \begin{bmatrix} A & BC_{wi} \\ 0 & A_{wi} \end{bmatrix} \qquad B_X = \begin{bmatrix} BD_{wi} \\ B_{wi} \end{bmatrix} \\ A_Y = \begin{bmatrix} A & 0 \\ B_{wo}C & A_{wo} \end{bmatrix} \qquad C_Y = [D_{wo}C \quad C_{wo}] \end{gathered} \tag{3}$$

Further, if A, A_{wi}, and A_{wo} are asymptotically stable, then X and Y are the solutions to

$$\begin{gathered} A_X X + X A_X^* + B_X B_X^* = 0 \\ A_Y^* Y + Y A_Y + C_Y^* C_Y = 0 \end{gathered} \tag{4}$$

FWIB states $\hat{x}$ are related to the state vector x in Eq. (1) by the transformation[5]

$$x(t) = VW\hat{x}(t) \tag{5}$$

where V decomposes $X_{11}Y_{11}$, X_{11}, and Y_{11} as

$$\begin{gathered} X_{11}Y_{11} = V\Sigma^2V^{-1} \qquad X_{11} = V\Sigma_c^2V^T \qquad Y_{11} = V^{-1^T}\Sigma_0^2V^{-1} \\ \Sigma = \Sigma_c\Sigma_0 \\ \Sigma = \mathrm{diag}\{\sigma_i\} \qquad \sigma_i \geq 0 \\ \Sigma_c = \mathrm{diag}\{\sigma_{c_i}\} \qquad \sigma_{c_i} \geq 0 \\ \Sigma_0 = \mathrm{diag}\{\sigma_{0_i}\} \qquad \sigma_{0_i} \geq 0 \end{gathered} \tag{6}$$

and W is defined as

$$W = \mathrm{diag}\{w_i\} \qquad w_i = \begin{cases} (\sigma_{c_i}/\sigma_{0_i})^{1/2} & \text{if} \quad \sigma_{c_i} \neq 0 \text{ and } \sigma_{0_i} \neq 0 \\ 1 & \text{if} \quad \sigma_{c_i} = 0 \text{ or } \sigma_{0_i} = 0 \end{cases} \tag{7}$$

Finally observe that X_{11} and Y_{11} are transformed such that $\hat{X}_{11} = \hat{Y}_{11} = \Sigma$.

Now assume the higher order model in Eq. (1) is FWIB and suppose the higher order and reduced-order models have dynamic order n and n_R, respectively. Further, suppose the σ_i are ordered from smallest to largest. Partitioning of $X_{11}Y_{11}$ as

$$\begin{gathered} X_{11}Y_{11} = \Sigma^2 = \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & \Sigma_2^2 \end{bmatrix} \\ \Sigma_1 = \mathrm{diag}\{\sigma_i\} \qquad i = 1,\ldots,n-n_R \\ \Sigma_2 = \mathrm{diag}\{\sigma_i\} \qquad i = n-n_R+1,\ldots,n \\ 0 \leq \sigma_1 \leq \cdots \leq \sigma_n \end{gathered} \tag{8}$$

leads to a partitioning of Eq. (1), or

$$\begin{gathered} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u(t) \\ y(t) = [C_1 \quad C_2]\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t) \end{gathered} \tag{9}$$

Truncation of the states x_1 leads to the reduced model[5]

$$\dot{x}_2(t) = A_{22}x_2(t) + B_2u(t) \qquad y(t) = C_2x_2(t) + Du(t) \tag{10}$$

Truncation Frequency Response Error

In this and the following sections, the first goal of developing an error bound for FWIB truncation is considered. Denote $G(s)$ and $G_R(s)$ as the higher order and reduced-order transfer function matrices corresponding to Eqs. (1) and (10), while $G_{wi}(s)$ and $G_{wo}(s)$ represent the input and output weighting transfer matrices corresponding to Eq. (2). The order reduction error is given as

$$E(s) = G(s) - G_R(s) \tag{11}$$

and the magnitude of the individual elements of $E(j\omega)$ in the frequency range of interest are an important measure of accuracy in the reduced-order model.[5,8,10,11,13] A closely related measure is the individual elements of the weighted frequency response error defined as

$$E_w(j\omega) = G_{wo}(j\omega)E(j\omega)G_{wi}(j\omega) \tag{12}$$

Note that if $G_{wi}(s)$ and $G_{wo}(s)$ leave the frequency response unaffected in the frequency range of interest and provide high attenuation otherwise, then $E_w(j\omega)$ and $E(j\omega)$ are essentially the same in the frequency range of interest, and $E_w(j\omega)$ is small otherwise. Finally, the maximum singular value of E_w, denoted as $\bar{\sigma}[E_w]$, is an upper bound on the magnitude of the elements of $E_w(j\omega)$.[9]

It can be shown that the weighted truncation error can be expressed as[5,11]

$$E_w(j\omega) = \tilde{C}_w(j\omega)\Delta^{-1}(j\omega)\tilde{B}_w(j\omega) \tag{13}$$

where

$$\begin{gathered} \tilde{B}_w(j\omega) = \tilde{B}(j\omega)G_{wi}(j\omega) \qquad \tilde{C}_w(j\omega) = G_{wo}(j\omega)\tilde{C}(j\omega) \\ \tilde{B}(j\omega) = B_1 + A_{12}\Phi(j\omega)B_2 \qquad \tilde{C}(j\omega) = C_1 + C_2\Phi(j\omega)A_{21} \\ \Phi(j\omega) = (j\omega I - A_{22})^{-1} \\ \Delta(j\omega) = j\omega I - A_{11} - A_{12}\Phi(j\omega)A_{21} \end{gathered} \tag{14}$$

Further, the maximum singular value of the weighted error can be expressed as

$$\bar{\sigma}^2[E_w] = \bar{\lambda}\left[\Delta^{-1}\tilde{B}_w\tilde{B}_w^*\Delta^{-1^*}\tilde{C}_w^*\tilde{C}_w\right] \tag{15}$$

where $\bar{\lambda}$ denotes the maximum eigenvalue. By expanding the products $\tilde{B}_w(j\omega)\tilde{B}_w^*(j\omega)$ and $\tilde{C}_w^*(j\omega)\tilde{C}_w(j\omega)$ and using the transformed, partitioned Eq. (4), it can be shown that[5,11]

$$\begin{gathered} \tilde{B}_w\tilde{B}_w^* = \Delta\left[\Sigma_1 + (X_{12})_1(-j\omega I - A_{wi}^*)^{-1}C_{wi}^*\tilde{B}^*\right] \\ + \left[\Sigma_1 + \tilde{B}C_{wi}(j\omega I - A_{wi})^{-1}(X_{21})_1\right]\Delta^* \\ \tilde{C}_w^*\tilde{C}_w = \Delta^*\left[\Sigma_1 + (Y_{12})_1(j\omega I - A_{wo})^{-1}B_{wo}\tilde{C}\right] \\ + \left[\Sigma_1 + \tilde{C}^*B_{wo}^*(-j\omega I - A_{wo}^*)^{-1}(Y_{21})_1\right]\Delta \end{gathered} \tag{16}$$

where $(X_{21})_1$ and $(Y_{12})_1$ are partitions of X_{21} and Y_{12}, respectively, induced by Eq. (9), or

$$X_{21} = [(X_{21})_1 \quad (X_{21})_2] \qquad Y_{12} = \begin{bmatrix} (Y_{12})_1 \\ (Y_{12})_2 \end{bmatrix} \tag{17}$$

After substituting Eq. (16) into Eq. (15), $\bar{\sigma}[E_w(j\omega)]$ can be rewritten as[5,11]

$$\begin{gathered} \bar{\sigma}^2[E_w] = \bar{\lambda}\Big[\left\{\Sigma_1 + M^* + \Delta^{-1}(\Sigma_1 + M)\Delta^*\right\} \\ \times\left\{\Sigma_1 + N + \Delta^{-1^*}(\Sigma_1 + N^*)\Delta\right\}\Big] \end{gathered} \tag{18}$$

where

$$\begin{gathered} M(j\omega) = \tilde{B}(j\omega)C_{wi}(j\omega I - A_{wi})^{-1}(X_{21})_1 \\ N(j\omega) = (Y_{12})_1(j\omega I - A_{wo})^{-1}B_{wo}\tilde{C}(j\omega) \end{gathered} \tag{19}$$

Note the $\Sigma_1 + M(j\omega)$ and $\Sigma_1 + N(j\omega)$ structure in Eq. (18), which is convenient for further error-bound simplification.

Truncation Bound for Order Reduction by One State

For $n - n_R = 1$, $\Sigma_1 = \sigma_1$, $\Delta(j\omega) = \delta(j\omega)$, $M(j\omega) = m(j\omega)$, and $N(j\omega) = n(j\omega)$ all become scalars, and Eq. (18) becomes[11]

$$\bar{\sigma}^2[E_w] = (1 + b^{-1}b^*a)(1 + c^{-1}c^*a^*)(\sigma_1 + m^*)(\sigma_1 + n) \tag{20}$$

where

$$a(j\omega) = \delta^{-1}(j\omega)\delta^*(j\omega)$$
$$b(j\omega) = \sigma_1 + m^*(j\omega) \qquad c(j\omega) = \sigma_1 + n(j\omega) \tag{21}$$

provided $\sigma_1 \neq -m^*(j\omega)$ and $\sigma_1 \neq -n(j\omega)$ for all frequencies. Note that σ_1 is a real number, whereas $m(j\omega)$ and $n(j\omega)$ are complex numbers, making it unlikely that $\sigma_1 = -m^*(j\omega)$ and $\sigma_1 = -n(j\omega)$. Since the right-hand side of Eq. (20) originates from a positive semidefinite matrix (i.e., $E_w E_w^*$), taking the absolute value does not alter the equality, or

$$\bar{\sigma}^2[E_w] = |1 + b^{-1}b^*a|\,|1 + c^{-1}c^*a^*|\,|\sigma_1 + m|\,|\sigma_1 + n|$$
$$\leq (1 + |b^{-1}b^*|\,|a|)(1 + |c^{-1}c^*|\,|a|)|\sigma_1 + m|\,|\sigma_1 + n| \tag{22}$$

Substitution of the equalities $|a(j\omega)| = 1$, $|b^{-1}(j\omega)b^*(j\omega)| = 1$, and $|c^{-1}(j\omega)c^*(j\omega)| = 1$ into Eq. (22) yields the error bound for order reduction by one state:[11]

$$\bar{\sigma}[E_w(j\omega)] \leq 2[|\sigma_1 + m(j\omega)| \bullet |\sigma_1 + n(j\omega)|]^{1/2} \tag{23}$$

Observe that the structure of this error bound is quite similar to that for IB truncation[5] and clearly reduces to the IB result when the weighting filters are selected as unity [i.e., $m(j\omega) = n(j\omega) = 0$].

Approximate Truncation Bound for General Case

Unfortunately, the error bound for order reduction by one state cannot be applied successively (without approximation) to obtain a useful error bound for the general case of order reduction by more than one state. This is because the reduced-order model from truncation is not FWIB. This is seen from the transformed, partitioned Eq. (4), or

$$A_{X_R}X_R + X_R A_{X_R}^* + B_{X_R}B_{X_R}^* = R_X$$
$$A_{Y_R}^* Y_R + Y_R A_{Y_R} + C_{Y_R}^* C_{Y_R} = R_Y \tag{24}$$

where

$$A_{X_R} = \begin{bmatrix} A_{22} & B_2 C_{wi} \\ 0 & A_{wi} \end{bmatrix} \qquad B_{X_R} = \begin{bmatrix} B_2 D_{wi} \\ B_{wi} \end{bmatrix}$$
$$A_{Y_R} = \begin{bmatrix} A_{22} & 0 \\ B_{wo}C_2 & A_{wo} \end{bmatrix} \qquad C_{Y_R} = [D_{wo}C_2 \quad C_{wo}]$$
$$X_R = \begin{bmatrix} \Sigma_2 & (X_{12})_2 \\ (X_{21})_2 & X_{22} \end{bmatrix} \qquad Y_R = \begin{bmatrix} \Sigma_2 & (Y_{12})_2 \\ (Y_{21})_2 & Y_{22} \end{bmatrix} \tag{25}$$
$$R_X = \begin{bmatrix} 0 & -A_{21}(X_{12})_1 \\ -(X_{21})_1 A_{21}^* & 0 \end{bmatrix}$$
$$R_Y = \begin{bmatrix} 0 & -A_{12}^*(Y_{12})_1 \\ -(Y_{21})_1 A_{12} & 0 \end{bmatrix}$$

For the reduced-order model to be FWIB, the residual terms R_X and R_Y would have to equal zero, and they are clearly not zero, in general.

Suppose for now that R_X and R_Y are small and negligible (this assumption is addressed later). Denote by $m_i(j\omega)$ and $n_i(j\omega)$ the variables corresponding to $m(j\omega)$ and $n(j\omega)$, respectively, in Eq. (19) for successive order reductions by one state. Also denote by $E_{w_i}(j\omega)$ the weighted frequency response error for each successive order reduction by one state, or

$$\bar{\sigma}[E_{W_i}(j\omega)] \leq 2[|\sigma_i + m_i(j\omega)| \bullet |\sigma_i + n_i(j\omega)|]^{1/2}$$
$$\text{for} \quad i = 1, \ldots, n - n_R \tag{26}$$

Now, the frequency response error $E_w(j\omega)$, for the general case of a reduction from n to n_R in one step, is related to the errors $E_{w_i}(j\omega)$ by

$$E_w(j\omega) = \sum_{i=1}^{n-n_R} E_{w_i}(j\omega) \tag{27}$$

Taking the singular value of Eq. (27) and pulling the summation outside the singular value yields

$$\bar{\sigma}[E_w(j\omega)] \leq \sum_{i=1}^{n-n_R} \bar{\sigma}[E_{w_i}(j\omega)] \tag{28}$$

Finally, substitution of Eq. (26) into Eq. (28) leads to the approximate error bound for the general case, or[11,17]

$$\bar{\sigma}[E_w(j\omega)] \leq 2\sum_{i=1}^{n-n_R}[|\sigma_i + m_i(j\omega)| \bullet |\sigma_i + n_i(j\omega)|]^{1/2} \tag{29}$$

Again observe the similarity with the error bound for IB truncation,[5] and note the result reduces to the IB bound when the weighting filters are selected as unity [(i.e., $m_i(j\omega) = n_i(j\omega) = 0$].

Observations Supporting FWIB Truncation Technique

Observe that the error bounds given in Eqs. (23) and (29) consist of two parts: the controllability-observability measures σ_i and the additional variables m_i and n_i. Clearly, if states with small values for σ_i are truncated, then the contribution to the error bound from σ_i will be correspondingly small. Of concern now is the significance of the truncated m_i and n_i to the error bound. Numerous examples[8,12–14] demonstrating that FWIB truncation yields an accurate reduced-order model in the frequency range of interest suggest that, if the truncated measures σ_i are sufficiently small, then the corresponding values for m_i and n_i and their contribution to the error bound will also be small. Support for this claim is considered next.

Let $\bar{X}(j\omega)$ and $\bar{Y}(j\omega)$ from Eq. (3) be partitioned as

$$\bar{X}(j\omega) = \begin{bmatrix} \bar{X}_G(j\omega)G_{wi}(j\omega) \\ \bar{X}_{wi}(j\omega) \end{bmatrix}$$
$$\bar{Y}(j\omega) = [G_{wo}(j\omega)\bar{Y}_G(j\omega) \qquad \bar{Y}_{wo}(j\omega)] \tag{30}$$

with

$$\bar{X}_G(j\omega) = (j\omega I - A)^{-1}B \qquad \bar{Y}_G(j\omega) = C(j\omega I - A)^{-1}$$
$$\bar{X}_{wi}(j\omega) = (j\omega I - A_{wi})^{-1}B_{wi} \quad \bar{Y}_{wo}(j\omega) = C_{wo}(j\omega I - A_{wo})^{-1} \tag{31}$$

Further, partition $\bar{X}_G(j\omega)$, $\bar{Y}_G(j\omega)$, $\bar{X}_{wi}(j\omega)$, and $\bar{Y}_{wo}(j\omega)$ as

$$\bar{X}_G(j\omega) = \begin{bmatrix} \bar{X}_{G_1}(j\omega) \\ \vdots \\ \bar{X}_{G_n}(j\omega) \end{bmatrix}$$
$$\bar{Y}_G(j\omega) = \left[\bar{Y}_{G_1}(j\omega) \quad \cdots \quad \bar{Y}_{G_n}(j\omega)\right] \tag{32}$$
$$\bar{X}_{wi}(j\omega) = \begin{bmatrix} \bar{X}_{wi_1}(j\omega) \\ \vdots \\ \bar{X}_{wi_p}(j\omega) \end{bmatrix}$$
$$\bar{Y}_{wo}(j\omega) = \left[\bar{Y}_{wo_1}(j\omega) \quad \cdots \quad \bar{Y}_{wo_q}(j\omega)\right]$$

where p and q denote the dynamic order of $G_{wi}(s)$ and $G_{wo}(s)$, respectively.

Using the notation in Eq. (30) and the transformed Eq. (3) yields

$$\Sigma = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{X}_G(j\omega)G_{wi}(j\omega)G_{wi}^*(j\omega)\bar{X}_G^*(j\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{Y}_G^*(j\omega)G_{wo}^*(j\omega)G_{wo}(j\omega)\bar{Y}_G(j\omega)\,d\omega \tag{33}$$

$$X_{21} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{X}_{wi}(j\omega)G_{wi}^*(j\omega)\bar{X}_G^*(j\omega)\,d\omega$$

$$Y_{12} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{Y}_G^*(j\omega)G_{wo}^*(j\omega)\bar{Y}_{wo}(j\omega)\,d\omega$$

Finally, from Eq. (33) and using the notation in Eq. (32),

$$\sigma_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\|\bar{X}_{G_i}(j\omega)G_{wi}(j\omega)\right\|_2^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\|G_{wo}(j\omega)\bar{Y}_{G_i}(j\omega)\right\|_2^2 d\omega \tag{34}$$

$$|(X_{21})_{1_{ij}}| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\|\bar{X}_{wi_i}(j\omega)\right\|_2 \left\|\bar{X}_{G_j}(j\omega)G_{wi}(j\omega)\right\|_2 d\omega \tag{35}$$

$$|(Y_{12})_{1_{ij}}| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\|G_{wo}(j\omega)\bar{Y}_{G_i}(j\omega)\right\|_2 \left\|\bar{Y}_{wo_j}(j\omega)\right\|_2 d\omega \tag{36}$$

Equations (34–36) are the key result here. First, observe from Eq. (34) that, for the smaller values of σ_i, one can expect $\|\bar{X}_{G_j}(j\omega)G_{wi}(j\omega)\|_2$ and $\|G_{wo}(j\omega)\bar{Y}_{G_i}(j\omega)\|_2$ in Eqs. (35) and (36) to be small, since the integrand in Eq. (34) is always nonnegative. Second, note from Eqs. (35) and (36) that $\|\bar{X}_{G_j}(j\omega)G_{wi}(j\omega)\|_2$ and $\|G_{wo}(j\omega)\bar{Y}_{G_i}(j\omega)\|_2$ directly contribute to upper bounds on the magnitude of the individual elements of $(X_{21})_1$ and $(Y_{12})_1$. Finally, recall that each element of $M(j\omega)$ and $N(j\omega)$ depends upon the elements of $(X_{21})_1$ and $(Y_{12})_1$.

Based upon the above observations, small values for the σ_i measures imply small numerical values for the elements of $M(j\omega)$ and $N(j\omega)$ and hence for m_i and n_i. Therefore, the contribution $|\sigma_i + m_i(j\omega)|\ |\sigma_i + n_i(j\omega)|$ to the upper bound on the weighted frequency response error in Eqs. (23) and (29) will be small when the corresponding measure σ_i is small. This establishes support for the FWIB truncation algorithm.

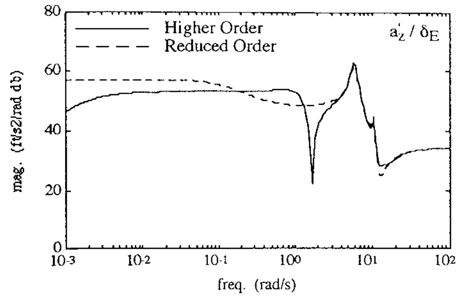
Returning to the assumption that the residual terms in Eq. (24) are small and negligible, note that the nonzero blocks of R_X and R_Y are direct functions of $(X_{21})_1$ and $(Y_{12})_1$. Then, based upon the previous development, a model obtained by truncating states having numerically small values for σ_i is nearly FWIB. The error bound in Eq. (29) for the general case of order reduction by more than one state is denoted approximate, in this sense.

As a final observation, Eqs. (23) and (29) raise an interesting question. Should states corresponding to the smallest values for $|\sigma_i + m_i\,(j\omega)|\ |\,\sigma_i + n_i(j\omega)|$ be truncated, rather than an algorithm based upon σ_i alone? This opens an entirely new avenue for research.

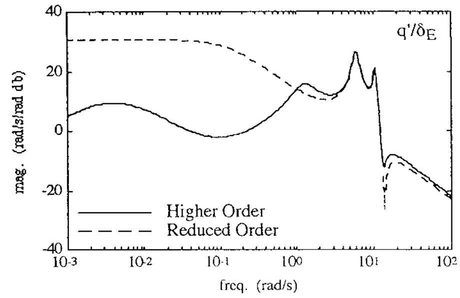
Truncation Example

Consider the model given in Refs. 11 and 17 describing the stable longitudinal dynamics of a large, flexible aircraft. The model is 12th order with phugoid, short period, and four aeroelastic modes. Control inputs consist of elevator deflection δ_E and canard deflection δ_C, whereas responses of interest are pitch rate q' and vertical acceleration a_z' attained from sensors located near the cockpit.

Suppose an accurate reduced-order model is desired in the frequency range above 3 rad/s, for structural mode control law development,[20] for example. A fifth-order model is obtained by FWIB truncation using an input weighting filter with unity magnitude above 3 rad/s and 40 db/dec attenuation below 3 rad/s. The frequency responses of the reduced-order and higher order models are shown in Figs. 1 and 2. Observe that the fifth-order model accurately reflects the dynamics of the higher order model in the frequency range of interest, above 3 rad/s.



a)



b)

Fig. 1 Elevator excited-frequency responses: truncation example.

To investigate the assertion that a reduced-order model, obtained by the elimination of states corresponding to small σ_i, is nearly FWIB, consider the following from Eq. (24):

$$\begin{aligned} &A_{22}\Sigma_2 + \Sigma_2 A_{22}^* + B_2 C_{wi}(X_{21})_2 + (X_{12})_2 C_{wi}^* B_2^* \\ &\quad + B_2 D_{wi} D_{wi}^* B_2^* = 0 \\ &A_{22}(X_{12})_2 + B_2 C_{wi} X_{22} + (X_{12})_2 A_{wi}^* + B_2 D_{wi} B_{wi}^* \\ &\quad = -A_{21}(X_{12})_1 \\ &A_{wi}(X_{21})_2 + X_{22} C_{wi}^* B_2^* + (X_{21})_2 A_{22}^* + B_{wi} D_{wi}^* B_2^* \\ &\quad = -(X_{21})_1 A_{21}^* \\ &A_{wi} X_{22} + X_{22} A_{wi}^* + B_{wi} B_{wi}^* = 0 \end{aligned} \tag{37}$$

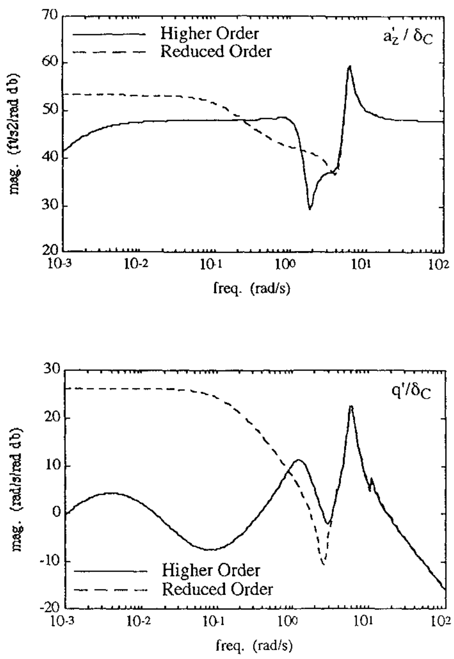
The second and third equations in Eq. (37) are the only equations that indicate a nonzero residual term. Further, the third equation is simply the complex conjugate transpose of the second equation; thus concentrate on the second equation only. Table 1 shows the maximum and minimum magnitudes over all the individual elements of the residual matrix $-A_{21}(X_{12})_1$ appearing in Eq. (37), as the model is reduced by one state at a time. Also shown are the corresponding values for n_R and σ_i. Note the correlation between small values for σ_i and small values for the elements of the residual matrix, making these reduced-order models essentially FWIB.

As an aside, observe the oscillatory behavior in the residual matrix maximum values as the reduction process proceeds as well as the clustering of the σ_i values. This behavior results from a complex-conjugate mode having to be split or approximated by a real mode, a difficult task indeed. In other words, if half of the complex mode is eliminated, then the other half might as well be eliminated.

Attention is now turned to the assertion that states corresponding to small σ_i inherently have small values for $m_i(j\omega)$ as well. Note $n_i(j\omega)$ equals zero for identity output weighting. Figure 3 shows the frequency responses of m_1 through m_7. Observe the correlation between small values for σ_i (see Table 1) and small values for m_i. Again, this is crucial to establishing support for the truncation technique, since σ_i and m_i contribute directly to an upper bound on the weighted, frequency-domain error.

Table 1 Residual term summary: truncation example

n_R	11	10	9	8	7	6	5
σ_i	0.0012 (σ_1)	0.084 (σ_2)	0.088(σ_3)	1.3 (σ_4)	4.8 (σ_5)	7.2 (σ_6)	45.(σ_7)
$\max_{i,j} \lvert\{-A_{21}(X_{12})_1\}_{ij}\rvert$	0.00015	0.14	0.0031	0.22	1.6	0.15	2.5
$\min_{i,j} \lvert\{-A_{21}(X_{12})_1\}_{ij}\rvert$	0.000000014	0.0000011	0.000012	0.00025	0.00046	0.0051	0.050



a)

b)

Fig. 2 Canard excited-frequency responses: truncation example.

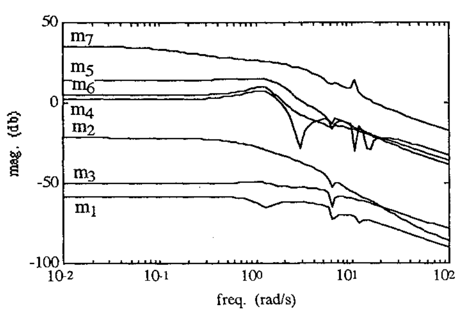


Fig. 3 Additional error bound term summary: truncation example.

Residualization with FWIB Coordinates

In this and the following sections, the second goal of estabilishing FWIB residualization as an equally valid reduction technique is considered. This primarily involves an error analysis, followed by numerical examples. Considerable development to follow parallels the analysis given in the previous sections for FWIB truncation. However, the reader is warned that the notation here represents features of the residualization technique, which are distinct from that of the truncation technique.

Recall the development of the weighted, balanced state-space realization in Eqs. (1–7). Suppose the σ_i are now ordered from largest to smallest. Partitioning of $X_{11}Y_{11}$ as

$$X_{11}Y_{11} = \Sigma^2 = \begin{bmatrix} \Sigma_1^2 & 0 \\ 0 & \Sigma_2^2 \end{bmatrix}$$
$$\Sigma_1 = \mathrm{diag}\{\sigma_i\} \qquad i = 1, \dots, n_R$$
$$\Sigma_2 = \mathrm{diag}\{\sigma_i\} \qquad i = n_R + 1, \dots, n \tag{38}$$
$$\sigma_1 \geq \cdots \geq \sigma_n \geq 0$$

also leads to the partitioning in Eq. (9). Residualization of the states x_2 (provided A_{22} is nonsingular) leads to the reduced model[11,18]

$$\begin{aligned} \dot{x}_1(t) &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)x_1(t) + \left(B_1 - A_{12}A_{22}^{-1}B_2\right)u(t) \\ y(t) &= \left(C_1 - C_2A_{22}^{-1}A_{21}\right)x_1(t) + \left(D - C_2A_{22}^{-1}B_2\right)u(t) \end{aligned} \tag{39}$$

Residualization Frequency Response Error

Here, the reduced-order model $G_R(s)$ corresponds to that in Eq. (39), rather than in Eq. (10). It can be shown that the weighted residualization error can be expressed as[11]

$$E_w(j\omega) = \tilde{C}_w(j\omega)[\Delta^{-1}(j\omega) - \Delta'^{-1}(j\omega)]\tilde{B}_w(j\omega) \tag{40}$$

where

$$\tilde{B}_w(j\omega) = \tilde{B}(j\omega)G_{wi}(j\omega) \qquad \tilde{C}_w(j\omega) = G_{wo}(j\omega)\tilde{C}(j\omega)$$
$$\tilde{B}(j\omega) = B_2 + A_{21}\Phi(j\omega)B_1 \qquad \tilde{C}(j\omega) = C_2 + C_1\Phi(j\omega)A_{12}$$
$$\Phi(j\omega) = (j\omega I - A_{11})^{-1} \tag{41}$$
$$\Delta(j\omega) = j\omega I - A_{22} - A_{21}\Phi(j\omega)A_{12}$$
$$\Delta'(j\omega) = -A_{22} - A_{21}\Phi(j\omega)A_{12}$$

Observe from Eq. (40) that, for zero ω, the weighted frequency response error resulting from this reduction technique is exactly zero. Further, the error is nearly zero in the low-frequency range where $j\omega$ is considered small relative to the diagonal elements of A_{22}, which represent the dynamics of the residualized coordinates. For truncation, the reverse characteristics hold, i.e., the error is small at high frequencies and precisely zero for infinite ω. Therefore, when FWIB reduction is being considered as a candidate technique, and the circumstances essentially call for elimination of high-frequency modes, residualization appears to be more appropriate than truncation. Further, if the specific problem calls for elimination of both high- and low-frequency modes, then a combination of FWIB residualization and FWIB truncation appears appropriate.

Now, the maximum singular value of the weighted error can be expressed as

$$\bar{\sigma}^2[E_w] = \bar{\lambda}\left[(\Delta^{-1} - \Delta'^{-1})\tilde{B}_w\tilde{B}_w^*(\Delta^{-1*} - \Delta'^{-1*})\tilde{C}_w^*\tilde{C}_w\right] \tag{42}$$

By expanding the products $\tilde{B}_w(j\omega)\tilde{B}_w^*(j\omega)$ and $\tilde{C}_w^*(j\omega)\tilde{C}_w(j\omega)$ and using the transformed, partitioned Eq. (4), it can be shown that[11]

$$\begin{aligned} \tilde{B}_w\tilde{B}_w^* &= \Delta\left[\Sigma_2 + (X_{12})_2(-j\omega I - A_{wi}^*)^{-1}C_{wi}^*\tilde{B}^*\right] \\ &\quad + \left[\Sigma_2 + \tilde{B}C_{wi}(j\omega I - A_{wi})^{-1}(X_{21})_2\right]\Delta^* \\ \tilde{C}_w^*\tilde{C}_w &= \Delta^*\left[\Sigma_2 + (Y_{12})_2(j\omega I - A_{wo})^{-1}B_{wo}\tilde{C}\right] \\ &\quad + \left[\Sigma_2 + \tilde{C}^*B_{wo}^*(-j\omega I - A_{wo}^*)^{-1}(Y_{21})_2\right]\Delta \end{aligned} \tag{43}$$

where $(X_{21})_2$ and $(Y_{12})_2$ are partitions of X_{21} and Y_{12} as in Eq. (17). After substituting Eq. (43) into Eq. (42), $\bar{\sigma}[E_w(j\omega)]$ can be rewritten as[11]

$$\begin{aligned} \bar{\sigma}^2[E_w] = \bar{\lambda}\Big[&\left\{\Delta^{-1}(\Sigma_2 + M)\Delta^* - \Delta'^{-1}(\Sigma_2 + M)\Delta'^* \right. \\ &\left. + j\omega\Delta'^{-1}(M - M^*)\right\}\left\{\Delta^{-1^*}(\Sigma_2 + N^*)\Delta \right. \\ &\left. - \Delta'^{-1^*}(\Sigma_2 + N^*)\Delta' + j\omega\Delta'^{-1^*}(N - N^*)\right\}\Big] \end{aligned} \tag{44}$$

where

$$M(j\omega) = \tilde{B}(j\omega)C_{wi}(j\omega I - A_{wi})^{-1}(X_{21})_2$$
$$N(j\omega) = (Y_{12})_2(j\omega I - A_{wo})^{-1}B_{wo}\tilde{C}(j\omega) \tag{45}$$

Note the $\Sigma_2 + M(j\omega)$ and $\Sigma_2 + N(j\omega)$ structure in Eq. (44), which is convenient for further error-bound simplification.

Residualization Bound for Order Reduction by One State

For $n - n_R = 1$, $\Sigma_2 = \sigma_n$, $\Delta(j\omega) = \delta(j\omega)$, $\Delta'(j\omega) = \delta'(j\omega)$, $M(j\omega) = m(j\omega)$, and $N(j\omega) = n(j\omega)$ all become scalars, and Eq. (44) yields[11]

$$\bar{\sigma}^2[E_w] = (a + p)(a^* + q)(\sigma_n + m)(\sigma_n + n^*) \tag{46}$$

where

$$a(j\omega) = \delta^{-1}(j\omega)\delta^*(j\omega) - \delta'^{-1}(j\omega)\delta'^*(j\omega)$$
$$p(j\omega) = j\omega\delta'^{-1}(j\omega)\{1 - [\sigma_n + m(j\omega)]^{-1}[\sigma_n + m^*(j\omega)]\}$$
$$q(j\omega) = j\omega\delta'^{-1^*}(j\omega)\{[\sigma_n + n^*(j\omega)]^{-1}[\sigma_n + n(j\omega)] - 1\} \tag{47}$$

provided $\sigma_n \neq -m(j\omega)$ and $\sigma_n \neq -n^*(j\omega)$ for all frequencies. Note that σ_n is a real number whereas $m(j\omega)$ and $n(j\omega)$ are complex numbers, making it unlikely that $\sigma_n = -m(j\omega)$ and $\sigma_n = -n^*(j\omega)$. Since the right-hand side of Eq. (46) originates from a positive semidefinite matrix (i.e., $E_w E_w^*$), taking the absolute value does not alter the equality, or

$$\bar{\sigma}^2[E_w] = |a + p|\,|a^* + q|\,|\sigma_n + m|\,|\sigma_n + n|$$
$$\leq (|a| + |p|)(|a| + |q|)|\sigma_n + m|\,|\sigma_n + n| \tag{48}$$

Substitution of the inequality $|a(j\omega)| \leq 2$ into Eq. (48) yields the error bound for order reduction by one state,[11]

$$\bar{\sigma}[E_w(j\omega)] \leq k(j\omega)[|\sigma_n + m(j\omega)|\,|\sigma_n + n(j\omega)|]^{1/2} \tag{49}$$

where

$$k(j\omega) = [(2 + |p(j\omega)|)(2 + |q(j\omega)|)]^{1/2} \tag{50}$$

Observe that the structure of this error bound is quite similar to that for IB residualization[18] and in fact reduces to the IB result when the weighting filters are selected to be unity [i.e., $m(j\omega) = n(j\omega) = 0$ and $p(j\omega) = q(j\omega) = 0$].

Note the scalar term $k(j\omega)$ multiplying the $|\sigma_n + m(j\omega)|\,|\sigma_n + n(j\omega)|$ term is now frequency dependent, rather than the constant value 2, as in Eq. (23). A direct analytical calculation of the maximum values of $|p(j\omega)|$ and $|q(j\omega)|$ is not practical; however, note the following results from Ref. 11. When ω tends to zero, $p(j\omega)$ and $q(j\omega)$ tend to zero, or

$$\lim_{\omega\to 0} p(j\omega) = 0 \qquad \lim_{\omega\to 0} q(j\omega) = 0 \tag{51}$$

When ω tends to infinity, $p(j\omega)$ and $q(j\omega)$ tend to the following:

$$\lim_{\omega\to\infty} p(j\omega) = 2 + \frac{B_2 D_{wi} D_{wi}^* B_2^*}{\sigma_n A_{22}}$$
$$\lim_{\omega\to\infty} q(j\omega) = -2 - \frac{C_2^* D_{wo}^* D_{wo} C_2}{\sigma_n A_{22}} \tag{52}$$

For intermediate values of ω, a direct numerical calculation of $p(j\omega)$ and $q(j\omega)$ for specific examples reveals that $|p(j\omega)|$ and $|q(j\omega)|$ are typically no larger than the limits in Eq. (52). Therefore, $k(j\omega)$ in Eq. (49) can be approximated by the constant value 4 (assuming D_{wi} and D_{wo} are zero), but here the error-bound result will be left in terms of $k(j\omega)$.

Approximate Residualization Bound for General Case

Unfortunately, as before, the error bound for order reduction by one state cannot be applied successively (without approximation) to obtain a useful error bound for the general case of order reduction by more than one state. This is because the reduced-order model from residualization is not FWIB. This is shown by multiplying the transformed, partitioned controllability Eq. (4) by Z_1 on the left and Z_1^* on the right, and multiplying the transformed, partitioned observability Eq. (4) by Z_2^* on the left and Z_2 on the right where

$$Z_1 = \begin{bmatrix} I & -A_{12}A_{22}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \qquad Z_2 = \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & 0 \\ 0 & I \end{bmatrix} \tag{53}$$

these operations lead to

$$A_{X_R}X_R + X_R A_{X_R}^* + B_{X_R}B_{X_R}^* = R_X$$
$$A_{Y_R}^* Y_R + Y_R A_{Y_R} + C_{Y_R}^* C_{Y_R} = R_Y \tag{54}$$

where

$$A_{X_R}\begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21}) & (B_1 - A_{12}A_{22}^{-1}B_2)C_{wi} \\ 0 & A_{wi} \end{bmatrix}$$

$$A_{Y_R} = \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21}) & 0 \\ B_{wo}(C_1 - C_2A_{22}^{-1}A_{21}) & A_{wo} \end{bmatrix}$$

$$B_{X_R} = \begin{bmatrix} (B_1 - A_{12}A_{22}^{-1}B_2)D_{wi} \\ B_{wi} \end{bmatrix}$$

$$C_{Y_R} = \begin{bmatrix} D_{wo}(C_1 - C_2A_{22}^{-1}A_{21}) & C_{wo} \end{bmatrix} \tag{55}$$

$$X_R = \begin{bmatrix} \Sigma_1 & (X_{12})_1 \\ (X_{21})_1 & X_{22} \end{bmatrix} \qquad Y_R = \begin{bmatrix} \Sigma_1 & (Y_{12})_1 \\ (Y_{21})_1 & Y_{22} \end{bmatrix}$$

$R_X =$

$$\begin{bmatrix} (B_1 - A_{12}A_{22}^{-1}B_2)C_{wi}(X_{21})_2A_{22}^{-1^*}A_{12}^* + A_{12}A_{22}^{-1}(X_{12})_2C_{wi}^*(B_1^* - B_2^*A_{22}^{-1^*}A_{12}^*) & A_{12}A_{22}^{-1}(X_{12})_2A_{wi}^* \\ A_{wi}(X_{21})_2A_{22}^{-1^*}A_{12}^* & 0 \end{bmatrix}$$

$R_Y =$

$$\begin{bmatrix} (C_1^* - A_{21}^*A_{22}^{-1^*}C_2^*)B_{wo}^*(Y_{21})_2A_{22}^{-1}A_{21} + A_{21}^*A_{22}^{-1^*}(Y_{12})_2B_{wo}(C_1 - C_2A_{22}^{-1}A_{21}) & A_{21}^*A_{21}^{-1^*}(Y_{12})_2A_{wo} \\ A_{wo}^*(Y_{21})_2A_{22}^{-1}A_{21} & 0 \end{bmatrix}$$

For the reduced-order model to be FWIB, the residual terms R_X and R_Y would have to equal zero and they are clearly not zero, in general.

Suppose for now that R_X and R_Y are small and negligible (this assumption is addressed later). Denote $m_i(j\omega)$, $n_i(j\omega)$, and $k_i(j\omega)$ as the variables corresponding to $m(j\omega)$, $n(j\omega)$, and $k(j\omega)$, respectively, in Eqs. (45) and (50) for successive order reductions by one state. Similar to the calculation given in Eqs. (26–29) for truncation, the approximate error bound for the general case is[11,17]

$$\bar{\sigma}[E_w(j\omega)] \leq \sum_{i=n_R+1}^{n} k_i(j\omega)[|\sigma_i + m_i(j\omega)| \bullet |\sigma_i + n_i(j\omega)|]^{1/2} \tag{56}$$

Again observe the similarity with the error bound for IB residualization,[18] and note the result reduces to the IB bound when the weightings are selected as unity [i.e., $m_i(j\omega) = n_i(j\omega) = 0$ and $k(j\omega) = 2$].

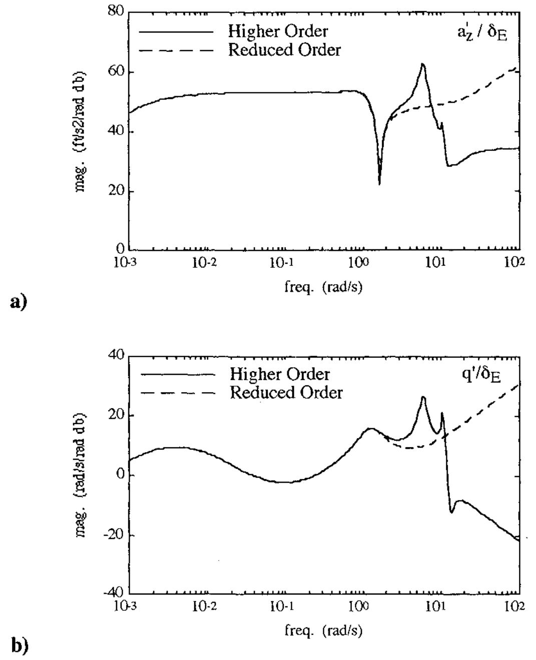


Fig. 4 Elevator excited-frequency responses: residualization example.

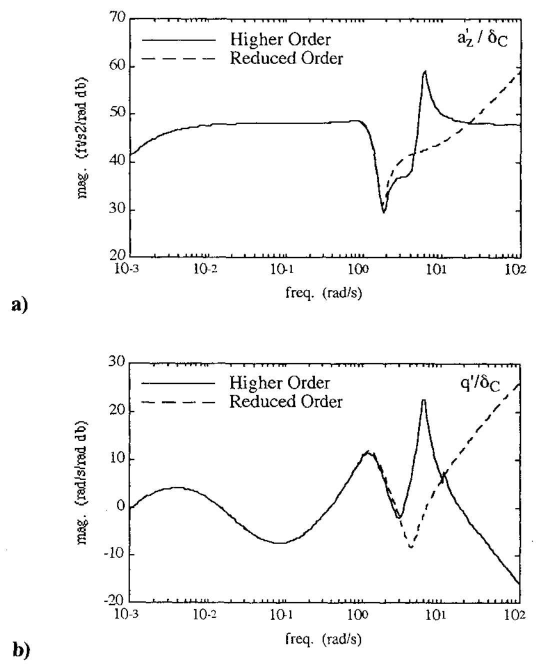


Fig. 5 Canard excited-frequency responses: residualization example.

Observations Supporting FWIB Residualization Technique

Observe that the error bounds given in Eqs. (49) and (56) again consist of two parts: the controllability-observability measures σ_i and the additional variables m_i and n_i. Suppressing the analysis similar to that given for truncation, it can be shown that residualized states with small values for σ_i will inherently possess small numerical values for the elements within $M(j\omega)$ and $N(j\omega)$ and hence for m_i and n_i. Therefore, the terms $|\sigma_i + m_i(j\omega)|\,|\sigma_i + n_i(j\omega)|$ appearing in the upper bound on the weighted frequency response error in Eqs. (49) and (56) will be correspondingly small. In other words, the weighted frequency response error will be small across the entire spectrum, and the unweighted frequency response error will be small in the desired region.

Returning to the assumption that the residual terms in Eq. (54) are small and negligible, note that the nonzero blocks of R_X and R_Y are direct functions of $(X_{21})_2$ and $(Y_{12})_2$. Thus, a reduced model, obtained by residualizing states having numerically small values for σ_i, is nearly FWIB. The error bound in Eq. (56) for the general case of order reduction by more than one state is denoted approximate, in this sense.

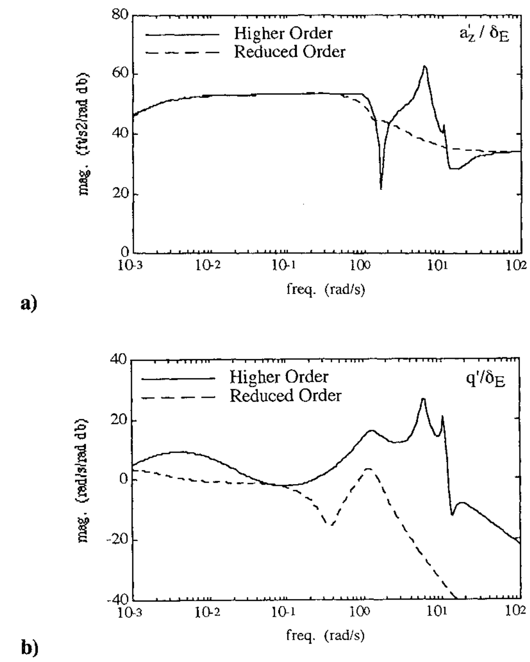


Fig. 6 Elevator excited-frequency responses using truncation: residualization example.

Residualization Example

Reconsider the model given in Refs. 11 and 17, and now suppose an accurate reduced-order model is required in the frequency range below 3 rad/s for an equivalent systems handling qualities analysis,[9] for example. A fifth-order model is obtained by FWIB residualization using an input weighting filter with unity magnitude below 0.5 rad/s and 40 db/dec attenuation above 0.5 rad/s. The frequency responses of the reduced-order and higher order models are shown in Figs. 4 and 5. Observe that the fifth-order model well approximates the dynamics of the higher order model in the frequency range of interest, below 3 rad/s.

To assess the usefulness of FWIB residualization, a fifth-order model obtained by FWIB truncation, using the same weighting filter as above, is shown in Figs. 6 and 7. It is clear that this fifth-order model does a poor job of predicting the dynamic characteristics of the higher order model in the frequency range of interest. Note with this technique the reduced-order model will never match, in general, the zero frequency gain of the higher order model. Although not shown, a sixth-order model obtained from FWIB truncation does a much better job of approximating the lower frequency range, with the trade-off of increased dynamic order.

Truncation-Residualization Example

To demonstrate that FWIB truncation and residualization may be used in a coordinated manner to achieve higher accuracy than that attainable from either technique used alone, suppose an accurate reduced-order model is desired in the 1–10 rad/s frequency range. This would be the case if the model were to be used for attitude flight control development,[21] for example. An input weighting filter with unity magnitude in the 1–10 rad/s frequency range and 40 db/dec attenuation otherwise is used. One fourth-order model is obtained by FWIB truncation. Another fourth-order model is obtained by a combination of FWIB truncation and residualization.

In the combined truncation-residualization technique with the σ_i ordered from smallest to largest as in Eq. (8), the states x_1 and x_6 are truncated and states x_2, x_3, x_4, x_5, x_7, and x_8 are residualized. By performing a modal analysis on the FWIB model, or by eliminating one state at a time and observing which mode is essentially

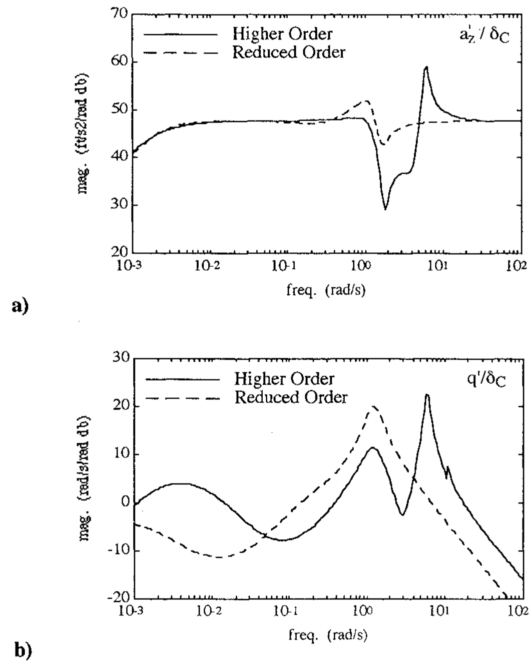


Fig. 7 Canard excited-frequency responses using truncation: residualization example.

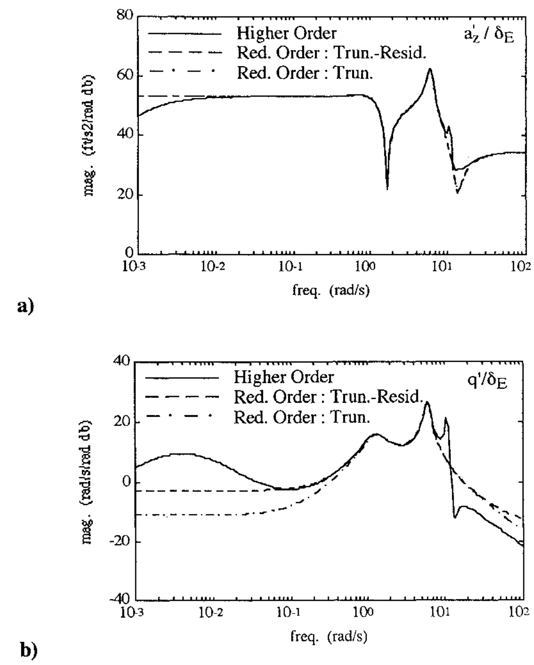


Fig. 8 Elevator excited frequency responses: combined truncation-residualization example.

eliminated, it can be found that states x_1 and x_6 are associated with a low-frequency (phugoid) mode, whereas states x_2, x_3, x_4, x_5, x_7, and x_8 are associated with modes at high frequency (second through fourth aeroelastic modes), relative to the frequency range of interest.

The frequency responses of the reduced-order and higher order models are shown in Figs. 8 and 9. As seen in Figs. 8 and 9, both fourth-order models accurately reflect the dynamics of the higher order model in the weighted frequency range as desired. However, note in the q'/δ_E and q'/δ_C frequency responses that the reduced-order model from FWIB truncation-residualization has improved accuracy in the 0.1–1-rad/s frequency range relative to the reduced-order model from FWIB truncation. Note this is achieved at the expense of less accuracy in the q'/δ_E frequency response around 40–100 rad/s. These results demonstrate that, even with FWIB reduction techniques, the user should still be aware of classical order reduction

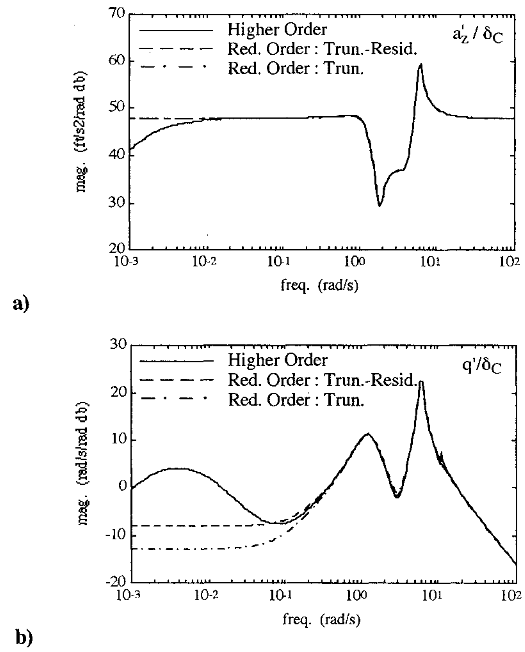


Fig. 9 Canard excited frequency responses: combined truncation-residualization example.

knowledge concerning truncation and residualization as well as the physics underlying the model.[13]

Conclusions

The real importance of a frequency response error analysis is not for an a priori assessment of the numerical reduction accuracy, but rather in gaining insight, offering guidance, and giving justification for the technique. In other words, the user understands how and why elimination of coordinates based upon a specific criterion leads to small frequency-domain error. The FWIB truncation error analysis presented here gives support for the truncation of coordinates based solely upon the weighted controllability-observability measures and explains how these measures contribute to an upper bound on the frequency response error. However, the error analysis results also suggest the possibility of increased accuracy in the reduced-order model if a new criterion is considered when selecting coordinates for elimination.

Residualization with FWIB coordinates has been shown to be a valid and appropriate order reduction technique when compared with FWIB truncation, especially when the problem fundamentally involves elimination of high-frequency modes. Also, a combined FWIB truncation-residualization procedure, consistent with classical truncation and residualization, can be used to obtain higher accuracy than that achievable from either technique used alone. Finally, new, multivariable, high-powered reduction techniques should not be applied blindly, but rather with caution, peppered with a classical understanding of order reduction and fundamental insight into the system dynamic characteristics.

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